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ABSTRACT

This document provides a preliminary report of the study Gateways To Advance Mathematical Thinking (GAMT) run by Educational Development Center, Inc. (EDC). The study was designed to see what types of reasoning students who have recently completed a linear algebra course apply to problems in algebraic thinking. Student interviews were used as the method of research with six questions on dot products, systems of linear equations, and linear transformations. In analyzing the data, six main issues were chosen as the focus of inquiry: dot product; relationship between a linear transformation and its matrix; solving a system of linear equations; meaning, number, and nature of the solution to a system of linear equations; visualization; and linear thinking: properties of a linear transformation. Preliminary results are based on early work with "dot product" and "meaning, number, and nature of the solution to a system of linear equations." Most of the paper consists of transcripts of interviews with students during their attempts to solve problems. Annotation of the interviews and analysis is also included. (MM)

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Student Learning in Linear Algebra: The Gateways to Advance Mathematical Thinking Project

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Education Development Center, Inc. is a private, non-profit education research and development company. EDC does research and development in many areas, including mathematics. In the past, EDC's mathematics work has been primarily at the K-12 level. In fact, in the past 6 years, projects have been funded to develop curricula at the elementary, middle, and high school levels. Recently we have begun doing work at the undergraduate level, and my presented paper is about one of these undergraduate projects: Gateways to Advance Mathematical Thinking.

The GAMT project is funded by the National Science Foundation. Our work is divided into three categories:

- Empirical research in student learning in Linear Algebra
- Curriculum development in precalculus
- Theoretical research on mathematical ways of thinking

The empirical research working group is looking at students' ways of reasoning about calculations and their use of "linear thinking." There are, we believe, ways of thinking in mathematics that are inherently algebraic, for example one can realize, without doing any calculations, that if one has two linear equations in two unknowns with integer coefficients, then the intersection must be at a rational point. We wanted to see what types of reasoning students who have recently completed a linear algebra course would apply to these types of problems.

We used student interviews as our method of research. We developed six questions in linear algebra; each of these questions looked like something you might see in a standard linear algebra class—including topics like dot products, systems of linear equations, and linear transformations—but they each were nonstandard in some way. We hoped these little "twists" in the problems would require students to reason about the problems rather than answering automatically. We tested these questions in three interviews, revised them based on the tests, and then conducted 12 more interviews with students from a private university in the Boston area. All of the students had recently completed a Linear Algebra course. Each interview was tape recorded and transcribed. The transcripts were coded for issues, and we compared issues within and across the interviews, deciding on six issues for analysis.

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In deciding which issues we would use for analysis, we had two criteria: there must be many entries for many students on that issue, and the entries must represent a range of success with the problems. The six issues we decided on are:

- dot product
- relationship between a linear transformation and its matrix
- solving a system of linear equations
- meaning, number, and nature of the solution to a system of linear equations
- visualization
- linear thinking: properties of a linear transformation

The results described in this paper are preliminary results based on our early work with "dot product" and "meaning, number, and nature of the solution to a system of linear equations." In both cases, we've found students using reasoning and problems solving tools specific to both of the issues, but we have also found more general tools that seem to cross issues, for example

- generalizing
- quoting a definition or theorem
- looking back at problems to get information or reassurance
- trying to rely on the way the problem is presented or worded

We have begun our analysis of students' use of generalizing in linear algebra problems. In the interviews, we have see students generalize after working with a unique example; need concrete examples, especially to check general results; refuse to generalize form one case; see the generality in one case; and generalize to higher dimensions. A few excerpts from the transcripts will demonstrate some of these uses of generalization. In each case, **S** refers to the student being interviewed, and **I** refers to the interviewer. Commentary for this paper is in italics.

Generalizing after working with a unique example

Here is the first problem from our interview. In the following excerpt, a student is trying to answer part d.

1. Suppose that $A = (1, 4, -2)$.

   (a) Find $A \cdot (4, 2, 0)$

   (b) Describe the set of all vectors X that have the property that $A \cdot X = 4$

   (c) If X is a vector so that $A \cdot X = 4$ and Y is a vector so that $A \cdot Y = 3$, what is $A \cdot (2X + 3Y)$? Explain.

   (d) If B and C are vectors so that $A \cdot B = A \cdot C$, what can you say about the relationship between B and C?

I: Right. What about going back to that parallel idea; I mean you had picked that for a reason. Obviously that came from something you knew or thought or ...worked out, or ...

The student had earlier said that B and C must be parallel if the equation in part d is true. The interviewer is reminding the student of this conjecture and asking for elaboration.

S: That, that had initially came to mind because I was thinking about the way that, um, you test for whether or not, vectors are perpendicular --- if the dot product is 0.

Then if, if, if uh the dot product of A and B was 0, and the dot product of A and C was 0, then you would have to say that ---B and C would have to be ---parallel.

The student selects a concrete example of when the dot products might be equal, one that the student is very familiar with, and tries to generalize from that situation.

I: Right. Right, cause A was perpendicular, you --- you know, to both.

S: But --- that's what I was just wondering is, about the same time, that if they were, just --- if it's just A dot B is equal to some constant, and A dot C is equal to the same constant, then, I would think that B and C may still have to be parallel, `cause it would be the same type of thing. I mean that you wouldn't be --- you would no longer have the relation that A and B were perpendicular to each other.

> But, there's still some fixed --- like if you were just talking about in 2D -- there's still some fixed angle between them. And there's still some fixed angle between the other one. So they would still be --- you would have a plane like this.

This is a perfectly reasonable thing for the student to have done. Unfortunately, the student chose a very special concrete example and came to a wrong generalization. Another student, using a very similar problem solving process, used the concrete example where both the dot products are equal to 4 (as in part b of this problem). Because the second student chose a "more general" specific example, he ended up solving the problem.

Working with a concrete example like this is one of the tools for the unconscious tendency to reduce the abstraction level. It is a result of the need to deal with something fan iliar, which you can work with, and not just treat its abstract properties. The main difference between working with a concrete example and the concept itself is that while one is working with a concrete example, he has something to "touch." When working with the concept, he actually has to work with a definition, and in many cases the students do not have the mental structure to connect with the abstract definition.

Needing concrete examples

In the next example, a student is answering part a of problem 2 from our interview, shown below. The student uses a concrete example to convince himself that his first instinct could not be right.

2. Consider the system of linear equations

$$\begin{aligned} 3x + 4y - 2z + w &= 6 \\ 5x - 2y + 2z + 2w &= 7 \\ -3x + 2y - z + 5w &= 3 \end{aligned}$$

(a) Without solving the system of equations, discuss how many solutions it might have?

(b) Find a solution to this system.

(c) Suppose you have two solutions to this system, say $A = (r_1, r_2, r_3, r_4)$ and $B = (s_1, s_2, s_3, s_4)$. Is $A + B$ a solution to the system? Why or why not?

(d) Write down a system of equations that has the property that if A is a solution, so is $3A$.

S: How many solutions might a 3-dimensional system in 4 unknowns have? Three equations, four unknowns how many solutions might have? Well... it could very easily have an infinite number of solutions, it could very easily have no solutions... I think. Maybe it can't have no solutions actually. In fact, no, there must be some solutions. If all three of these equations were independent you would still have at least an entire line of solutions. The way that I see that I think.... If I wrote down a matrix which is what I will do in part (b), you only have 3 rows and the whole concept of rank and all those things that linear algebra people like to deal with, but I don't like to talk about that...

The student is unsure of the possible numbers of solutions when thinking about an abstract system of three equations in four unknowns. He can see the potential for infinitely many, but is not sure if there can be no solutions.

[later]

I: At the beginning you wondered whether there should be a solution or if there's a case that there's no solution. Could you talk about this?

After the student solved this particular system, the interviewer asks him to explore that conjecture again.

S: My intuition at first is that there should be a way to get no solution and I didn't really think about it much, but if you think about it any time if you had four coefficients the same in two equations, if you had 3x+4y-2z+w=6 and then you had 3x+4y-2z+w=7 then if you think about it there's no x, y, z, and w such that 6=7. You can't do this, so it is possible if you have two completely contradictory equations is possible to have no solution again, so that intuition was... But I think the deal here is that is not possible to have one unique solution which is the beauty of n equations and n unknowns, if they are all independent you can get one unique solution. If there were no w here and you just had something x, y, z equals something and all the equations were independent then you would have one

So the student comes up with a very specific example where there is no solution to help him answer the question. Without this concrete example, he was having difficulty convincing himself one way or the other.

Seeing the generality in one case

A third student, answering part d of the question 2 (shown above), used a very concrete example to argue that the system he had developed was closed under scalar multiplication.

S: Because I plugged in -b for a which gives c=0. So a can be anything, b is always going to be the negative of a. And then c is always going to be 0. So we have (a,-a,0) as our solutions.

(a,-a,0) was the solution to the student's system of equations. This was the second set of equations the student had set up, the first being a heterogeneous system which he found failed the test that if A was a solution, so was 3A.

If a, this'll work, okay so if, if I multiply by 3, say a=1, b is going to be -1, c is always gonna be 0. If I multiply it by 3 you'll get (3,-3,0), which works. So it's closed under scalar multiplication.

The student chooses a specific a (a=1) to test, rather than working with his general solution of (a,-a,0). He performs the very specific test required: if A is a solution, so is 3A. But the conclusion he draws is a very general one: the system is closed under scalar multiplication. He sees the generality both in his choice of a and the problem's choice of scalar. He explains below what makes this system "work."

I: Why does it work? Why this one works and the, the previous one that you wrote doesn't work? Can think of a special reason?

S: Now that I'm thinking about it more I think I can. Because there's no constant value, there's no... It's not like the solution, if we look before we had 2-y. Which would be a constant minus whatever y is, okay. Our, our solution always has to be in the form of 2 minus that number, okay.

But here you have no constants to worry about when you're subtracting. So whatever the number is, b is always gonna be the negative of that number, okay. So it doesn't matter if we multiply by 3. If we had lets say a-1, when you multiply by 3, if, if our solution was in the form of (a-1,-a,0), if we multiplied this by 3 we're gonna get (3a-3,-3,0). And this -3 factor right here makes it not closed under scalar multiplication.

The student again uses 3 as a "general scalar" in an argument.

Refusing to generalize from one case

In contrast to this seeing generality while working with concrete examples, we have found students who refuse to generalize from specific cases. For example, one student was answering part c of our third question, shown below. The student noticed what he called a "pattern," but did not want to draw any hasty conclusions based on it.

3. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T[(x, y, z)] = (x + 2y - 3z, 2y - z)$$

(a) Find $T[(7, 3, 2)]$

(b) Find $T[(0, 0)]$

(c) If $X = (4, 1, 0)$, find $T[X]$. Find $T[2X]$.

(d) Find all X so that $T[X] = (-3, -7)$

(e) If $Y = (1, 0, 0)$, find all X such that $T[X] = Y$

(f) What would it mean to have a matrix A so that $T[(x, y, z)] = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$?

(g) Find such a matrix A.

S: If A= (4,1,0). So I'm going to do (4+2-0, 2-0) ... (6,2) ...T(A). 2A= (8,2,0). T(2A)=(8+4-0, 4-0). So we have (12,4).

The student mumbles a bit while computing T(A) and T(2A). To find T(2A), he first finds 2A, then applies the transformation T. This is a perfectly reasonable way to solve the problem, but the interviewer probes to find out if he realizes that he could also solve it by multiplying T(A) by 2.

I: So how did you do that?

S: I just; to find T(A) I just used the definition of T. The function the linear transformation T. I just plugged those numbers in and solved and to find T(2A) what I first did was I wrote down what 2A was equal to and then I plugged the value of 2A into my linear transformation and evaluated and I think that there is a pattern here.

I: Here, where?

S: Between A... T(A) and T(2A). It looks like T(2A) is 2 times T(A). So I would say that... This is just a hypothetical. I don't know. T(A)...T(2A) likes like 2T(A). Find b such that...

I: But your not sure of that.

S: Well I only have one case here and you should never make a generalization about one case.

The student has undoubtedly heard this quote many times, "You should never make a generalization about one case." And in a way he is right. But as Polya says, "Here is the mathematical aspect of generalization: Solving a simpler version of the proposed problem and then generalizing." When students do this, they are doing mathematics. They are also showing something of their grasp of linear algebra topics. Is the general notion of "vector" something students can comfortably work with and talk about, or do they need to write down and manipulate a specific vector? Generality and abstraction are key concepts in linear algebra, as in all of mathematics, but we must remember that students need time to think about things in the same way that mathematicians do.